

# The Energy-Momentum Tensor in the 1+1 dimensional non-rotating BTZ black hole

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## Abstract

We study the energy-momentum tensor for the real scalar field on the 1+1 dimensional non-rotating BTZ black hole. We obtain closed expressions for it.

## 1 Introduction

It is well-known that the energy-momentum tensor in Quantum Field Theory in curved spacetime is a subtle issue. This is principally due to the divergences which occur when the expectation value of it in a certain state is calculated, see for example [1] for an extensive discussion. However, since it contains important physical information of the field it is worth trying to calculate it. It turns out that in 1+1 dimensions most of the difficulties can be removed and it is possible to obtain closed expressions for it [2]. In this paper we will exploit this fact and will calculate this quantity for the 1+1 dimensional non-rotating BTZ black hole<sup>1</sup>. Even though a great amount of work on calculating the expectation value of the energy-momentum tensor in several two dimensional spacetimes has been done, see for example references in [2] and [5], as far

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<sup>1</sup>It is worth mentioning that there exist the rotating BTZ black hole in 1+1 dimensions too, see for example [3] and [4].

as we know the study of this tensor without walls in the 1+1 dimensional non-rotating BTZ black hole has not been done before<sup>2</sup>. The main motivation of this work is to fill this gap in the literature. Also it is worth mentioning that although the calculations are very simple interesting results are obtained and closed expressions as well.

## 2 The energy-momentum tensor in 1+1 dimensions

In 1+1 dimensions the energy-momentum tensor is almost determined by its trace. In what follows we give the basic formule for calculating this quantity.

Let us consider the following metric

$$ds^2 = C(-dt^2 + dx^2) = -Cdu dv, \quad (1)$$

where  $u = t - x$  and  $v = t + x$ . Since every 1+1 dimensional metric is conformal to a 1+1 dimensional metric in Minkowski spacetime, the metric (1) is very general. The function  $C$  in general depends on both variables in the metric. In these circumstances the expectation value of the trace of the energy-momentum is [2]

$$\langle T^\mu_\mu \rangle = -\frac{R}{24\pi} = \frac{1}{6\pi} \left( \frac{C_{uv}}{C^2} - \frac{C_u C_v}{C^3} \right), \quad (2)$$

where  $R$  is the Ricci scalar and  $C_u = \frac{\partial}{\partial u} C$ , etc. The last expression holds for the real scalar field. The expectation value of the components of the energy-momentum tensor in null coordinates is

$$\langle T_{uu} \rangle = -\frac{1}{12\pi} C^{1/2} \partial_u^2 C^{-1/2} + f(u) \quad (3)$$

$$\langle T_{vv} \rangle = -\frac{1}{12\pi} C^{1/2} \partial_v^2 C^{-1/2} + g(v) \quad (4)$$

where  $f$  and  $g$  are arbitrary functions of  $u$  and  $v$  respectively. These functions contain information about the state with respect to which the expectation value is taken. The mixed components are given by

$$\langle T_{uv} \rangle = -\frac{CR}{96\pi}. \quad (5)$$

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<sup>2</sup>A similar study of the energy-momentum tensor in the BTZ black hole in 1+1 dimensions with walls has been done in [6]. The results of this work can be considered complementary to the present work.

Now let us apply these formulae to the 1+1 dimensional non-rotating BTZ black hole.

## 2.1 The energy-momentum tensor in the 1+1 dimensional non-rotating BTZ black hole

The metric for the 1+1 dimensional non-rotating BTZ black hole can be written in the form (1) with

$$C = N^2 = \left( -M + \frac{r^2}{l^2} \right) \quad (6)$$

and  $r^* = x$  where  $r^*$  is the tortoise like coordinate defined by  $\frac{dr^*}{dr} = C^{-1}$ . The function  $C$  can be written as function of  $r^*$  or  $u$  and  $v$  as follows

$$C = \frac{M}{\sinh^2 \kappa r^*} = \frac{M}{\sinh^2 \kappa \frac{(v-u)}{2}}, \quad (7)$$

where  $\kappa = \frac{r_+}{l^2}$  is the surface gravity with  $r_+ = l\sqrt{M}$ .

Using the previous expression for  $C$  we obtain

$$\langle T_{uu} \rangle = -\frac{\kappa^2}{12\pi} + f(u) \quad (8)$$

and

$$\langle T_{vv} \rangle = -\frac{\kappa^2}{12\pi} + g(v). \quad (9)$$

Using that

$$T_{tt} = T_{uu} + 2T_{uv} + T_{vv}, \quad (10)$$

$$T_{xx} = T_{uu} - 2T_{uv} + T_{vv}, \quad (11)$$

and

$$T_{tx} = -T_{uu} + T_{vv} \quad (12)$$

we obtain

$$\langle T_{tt} \rangle = -\frac{\kappa^2}{6\pi} + \frac{CR}{48\pi} + f(u) + g(v), \quad (13)$$

$$\langle T_{xx} \rangle = -\frac{\kappa^2}{6\pi} - \frac{CR}{48\pi} + f(u) + g(v) \quad (14)$$

and

$$\langle T_{tx} \rangle = g(v) - f(u). \quad (15)$$

From the last three expressions it follows that

$$\langle T^\mu{}_\nu \rangle = A + B + D \quad (16)$$

where

$$A = \frac{\kappa^2}{6\pi C} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (17)$$

$$B = -\frac{R}{48\pi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (18)$$

$$D = \frac{1}{C} \begin{pmatrix} -f(u) - g(v) & f(u) - g(v) \\ g(v) - f(u) & f(u) + g(v) \end{pmatrix}. \quad (19)$$

If we choose  $f(u) = g(v) = 0$  we obtain the analogue of the Boulware state in Schwarzschild spacetime which is singular at the horizon ( $C = 0$ ). However we can also obtain the analogous of the Hartle-Hawking state which is regular in both the future and the past horizons. The value of  $f(u)$  and  $g(v)$  can be obtained if we express the energy-momentum tensor in Kruskal like coordinates,  $U$  and  $V$ . In these coordinates the energy momentum tensor is

$$\langle T_{UU} \rangle = \frac{1}{U^2} \langle T_{uu} \rangle = \frac{1}{U^2} \left( \frac{f(u)}{\kappa^2} - \frac{1}{12\pi} \right), \quad (20)$$

$$\langle T_{VV} \rangle = \frac{1}{V^2} \langle T_{vv} \rangle = \frac{1}{V^2} \left( \frac{g(v)}{\kappa^2} - \frac{1}{12\pi} \right), \quad (21)$$

and

$$\langle T_{UV} \rangle = 0. \quad (22)$$

Hence we demand that  $f(u) = g(v) = \frac{\kappa^2}{12\pi}$ , although it is not the only possibility. We could choose functions which close to the horizon have the values  $\frac{\kappa^2}{12\pi}$  and a different value far from it. So  $f$  and  $g$  constants are not the only possibility. It is worth pointing out that there is no natural analogue of the Unruh vacuum, since this would lead us to have no conservation of energy-momentum at infinity. Because of the timelike nature of this boundary we must impose no net flux of momentum at infinity and the Unruh state would violate this condition.

Now let us explain the physical interpretation for the functions  $f(u)$  and  $g(v)$ . From (19) we see that these functions are related with the flux of energy-momentum in the spacetime. For example, for the Boulware state we have no flux of momentum at infinity since  $C$  goes to infinity there and kills this flux momentum for any functions,

however for this state we have flux of energy-momentum at points in the exterior of the black hole. So in a sense the state is not static. Whereas for the Hartle-Hawking state there is no flux at no point of the black hole, so we have a, let us say, static state, we just have energy density and pressure at every point of the spacetime. So the functions  $f(u)$  and  $g(v)$  control the nature of the state on the black hole.

Also it is interesting to write this tensor in an orthonormal frame. This can be done by introducing two-beins. The appropriate orthonormal frame is given by

$$e^a_t = (N, 0) \quad (23)$$

and

$$e^b_r = (0, N), \quad (24)$$

where  $a$  and  $b$  are indexes associated with the orthonormal frame. In these circumstances the energy-momentum tensor is given by

$$\langle T^{ab} \rangle = E + F + G \quad (25)$$

where

$$E = -\frac{\kappa^2}{6\pi C} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (26)$$

$$F = \frac{R}{48\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (27)$$

$$G = \frac{1}{C} \begin{pmatrix} f(u) + g(v) & f(u) - g(v) \\ f(u) - g(v) & f(u) + g(v) \end{pmatrix} \quad (28)$$

From this expression we see that for the Hartle-Hawking state

$$\langle T^{ab} \rangle = \frac{R}{48\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (29)$$

Since in the present case<sup>3</sup>  $R = -\frac{2}{l^2}$ , then the energy density and the pressure are given respectively by

$$\rho = -\frac{1}{24l^2\pi} \quad (30)$$

and

$$p = \frac{1}{24l^2\pi}. \quad (31)$$

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<sup>3</sup>This is because locally the non-rotating BTZ black hole is AdS spacetime.

Hence the radius  $l$  determines the properties of the field in the 1+1 dimensional non-rotating BTZ black hole. It is clear from (30) that there is no flux of momentum at infinity however there is a different from zero energy density at every point of the spacetime.

It is also interesting to look at the semiclassical Einstein field equations. It is well known that in 1+1 dimensions the Einstein tensor vanishes identically, so there are no Einstein equations. In particular the Einstein field equations with cosmological constant in vacuum are impossible, however we now show that a kind of Einstein field equations with cosmological constant make sense when the right hand side of them is taken to be the expectation value of the energy-momentum tensor we have found.

If we write the semiclassical Einstein field equations as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \langle T_{\mu\nu} \rangle, \quad (32)$$

with  $\Lambda < 0$ , then in 1+1 dimensions the first two terms of the left hand side vanish identically and we are left with

$$\Lambda g_{\mu\nu} = \langle T_{\mu\nu} \rangle. \quad (33)$$

But according to our expressions for the energy-momentum tensor this equality can be satisfied if and only if

$$g_{\mu\nu} = \frac{CR}{48\pi\Lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (34)$$

which is no other thing than the metric for the BTZ black hole in 1+1 dimensions, scaled by an overall factor and multiplied by minus one. This could be interpreted as the metric inside the horizon. Hence we could say that the back reaction shifted the horizon by making it bigger. It is interesting to note that if the sign would be opposite then there would not be change in the geometry. Since if we take this metric as the starting point for calculating the expectation value of the energy-momentum tensor we would find the same values as previously [5]. In this second scenario the BTZ metric would be stable under back reaction effects. The expression (34) and the previous analysis could be suggestive and take  $\Lambda = \frac{R}{48\pi} = -\frac{1}{24\pi l^2}$  in two dimensions, so we would have an analogous expression for two dimensions to that of other dimensions where  $\Lambda = \frac{d(1-d)}{2l^2}$  and  $d+1$  is the dimension of the spacetime.

The previous discussion should be taken with care and just as an indication of the possible scenarios, since there is no Einstein equations in 1+1 dimensions. A more natural thing to do would be to plug in the expectation value of the energy-momentum we have found into a theory of gravity in 1+1 dimensions and see how it is modified.

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